Exercise 12

(a) Sodium chlorate crystals are easy to grow in the shape of cubes by allowing a solution of water and sodium chlorate to evaporate slowly. If V is the volume of such a cube with side length x, calculate dV/dx when $x = 3$ mm and explain its meaning.

(b) Show that the rate of change of the volume of a cube with respect to its edge length is equal to half the surface area of the cube. Explain geometrically why this result is true by arguing by analogy with Exercise 11(b).

Solution

Part (a)

Take the derivative of

$$V(x) = x^3$$

to find how the volume changes when the side length changes.

$$V'(x) = 3x^2$$

Consequently,

$$V'(3) = 3(3)^2 = 27 \text{ mm}^2.$$

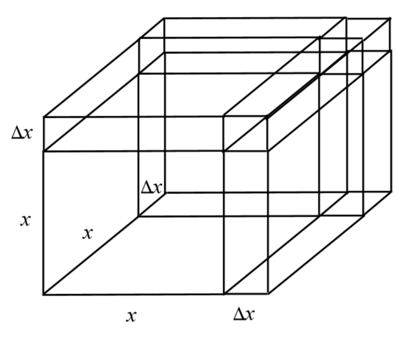
This means that when the side length is 3 mm, the volume is increasing by 27 mm^3 per millimeter of side length.

Part (b)

The surface of a cube consists of six faces each with area x^2, so the total surface area is $S = x^2 + x^2 + x^2 + x^2 + x^2 + x^2 = 6x^2$.

$$V'(x) = \frac{S}{2}.$$

Suppose there's a cube with side length x, and the side length then increases by Δx.



The old volume is $V_{\text{old}} = x^3$, and the new volume is

$$V_{\text{new}} = x^3 + x^2\Delta x + x^2\Delta x + x^2\Delta x + x\Delta x\Delta x + x\Delta x\Delta x + x\Delta x\Delta x + \Delta x\Delta x\Delta x$$

$$= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

Because Δx is assumed to be small, $3x(\Delta x)^2$ and $(\Delta x)^3$ are extremely small compared to $x^3 + 3x^2\Delta x$ and can be neglected to a good approximation.

$$V_{\text{new}} \approx x^3 + 3x^2\Delta x$$

Therefore, the approximate change in volume is

$$\begin{aligned}\Delta V &= V_{\text{new}} - V_{\text{old}} \\ &\approx (x^3 + 3x^2\Delta x) - x^3 \\ &\approx 3x^2\Delta x.\end{aligned}$$